

G-coupling functions

MORALES SILVA D.M.¹, RUBINOV A.M.² and SOSA W.³

¹ SITMS, University of Ballarat,
Victoria 3353, Australia. E-mail: 2555675@fs2.ballarat.edu.au

² SITMS, University of Ballarat,
Victoria 3353, Australia. E-mail: a.rubinov@ballarat.edu.au

³ IMCA, Universidad Nacional de Ingeniería,
Jirón Ancash 536, Lima 1, Lima, Perú, e-mail: sosa@uni.edu.pe

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GAP functions are useful for solving Optimization Problems, but the literature contains a variety of different concepts of GAP functions. It is interesting to point out that these concepts have many similarities. Here we introduce G-coupling functions, thus presenting a way to take advantage of these common properties.

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1 Introduction

For solving non-convex optimization problems, a tool that is becoming more important is **generalized conjugation**. In this paper we introduce a family of coupling functions, the G-coupling functions, which will allow us to see in a different way duality schemes. The usual properties found in the literature ([11], [13]) are related to a fixed coupling function, but here we consider (for a specified function f) a family of coupling functions.

These coupling functions are motivated by gap functions. It is interesting to point out, that many of these (gap) functions have similar properties. However, in some cases they are functions of one vector and it is important, since they are linked to specified optimization problems, that those functions have zeros.

G-coupling functions will be defined as functions in two variables and they might not have zeros. Even more, given a specified proper function f , it is shown that a sub-family of this family of coupling functions satisfies many

interesting properties.

In Section 2, we describe how many gap functions have similar properties, which are useful for the definition of G-coupling functions.

In Section 3, it is found the definition of G-coupling function with properties related to generalized conjugation using this family of functions and a fixed proper function f .

In Section 4, it can be seen how these ideas can be applied in the Equilibrium Problem.

2 Motivation

In several works already published, there can be found definitions of GAP functions for particular problems. Now we present 3 concrete examples.

In [4], it is consider the following Variational Inequality Problem:

$$(VIP) \text{ Find } x_0 \in C, \text{ such that, } \exists y^* \in T(x_0) \text{ with } \langle y^*, x - x_0 \rangle \geq 0 \quad \forall x \in C,$$

where T is a maximal monotone correspondence (i.e., $\langle u - v, x - y \rangle \geq 0$ for every $u \in T(x)$, $v \in T(y)$ with $x, y \in C$ and if there exists v , such that $\langle u - v, x - y \rangle \geq 0$, for all $x, y \in C$ and for all $u \in T(x)$, then $v \in T(y)$). It is found as a GAP function the following one:

$$h_{T,C}(x) := \sup_{(v,y) \in G_C(T)} \langle v, x - y \rangle,$$

where $G_C(T) = \{(v, y) : v \in T(y), y \in C\}$ and C is a non-empty closed convex set. This function happens to be non-negative and convex, and it is equal to zero only in solutions of (VIP).

The theory of the Equilibrium Problem begun with the paper written by Blum and Oettli [3]:

$$(EP) \text{ Find } x \in K, \text{ such that } f(x, y) \geq 0, \quad \forall y \in K,$$

where $K \subset \mathbb{R}^n$ is a non-empty closed convex set and $f : K \times K \rightarrow \mathbb{R}$ is a function that satisfies:

- i) $f(x, x) = 0$, for all $x \in K$.
- ii) $f(x, \cdot) : K \rightarrow \mathbb{R}$ is convex and l.s.c.
- iii) $f(\cdot, y) : K \rightarrow \mathbb{R}$ is u.s.c.

The GAP function is defined as ([8], [13]):

$$g_f(y) := \begin{cases} \sup_{x \in K} f(x, y) & \text{if } y \in K \\ +\infty & \text{in other case.} \end{cases}$$

In this case, the function g_f is non-negative, convex and l.s.c. and if it vanishes at x_0 , then x_0 is a solution of (EP).

In [16], the Extended Pre-Variational Inequality Problem is considered:

(EPVIP) Find $x_0 \in \mathbb{R}^n$, such that $\langle F(x_0), \eta(x, x_0) \rangle \geq f(x_0) - f(x)$, $\forall x \in \mathbb{R}^n$,

where $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$. In this work, the GAP function is

$$\min_{y \in \mathbb{R}^n} [\langle F(x), \eta(y, x) \rangle - f(x) + f(y)],$$

which is non-positive and it only reaches the value zero in solutions of (EPVIP).

In all these examples, gap functions are used to transform an special Equilibrium Problem (for example the VIP is a particular class of EP) into a minimization problem.

Now our attention is focused in using coupling functions that could be related, at least in some general aspect, to GAP functions. Therefore these functions must link both primal and dual variables. Since these coupling functions must be related to a sense of “gap”, we consider these functions as non-negative and with 2 arguments.

Let us remember that for the minimization problem, the convex conjugation theory allows us to generate a dual problem and there is implicit another concept of gap function (see [6]): consider

$$\alpha = \inf[f(x) : x \in \mathbb{R}^n]. \quad (P)$$

Define a function $\varphi : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, satisfying

$$\varphi(x, 0) = f(x), \quad \forall x \in \mathbb{R}^n.$$

Then φ will be called a perturbation function and the function $h : \mathbb{R}^p \rightarrow \overline{\mathbb{R}}$ defined by

$$h(u) = \inf_{x \in \mathbb{R}^n} \varphi(x, u)$$

will be called the marginal function. Observe that

$$\alpha = h(0) = \inf_{x \in \mathbb{R}^n} \varphi(x, 0) = \inf_{x \in \mathbb{R}^n} f(x).$$

Considering now h^{**} , the convex bi-conjugate (see [10]) of h one has:

$$h^{**}(0) \leq h(0) = \alpha$$

where

$$h^{**}(0) = \sup[\langle u^*, 0 \rangle - h^*(u^*) : u^* \in \mathbb{R}^p].$$

Then, making $-\beta = h^{**}(0)$, one has

$$\beta = \inf_{u^* \in \mathbb{R}^p} h^*(u^*). \quad (Q)$$

(Q) is called dual problem of (P) and in general we have $-\beta \leq \alpha$. It is said that there is no duality gap whenever $h^{**}(0) = h(0)$. It is easy to prove that $h^*(u^*) = \varphi^*(0, u^*)$, and if we define the function $k : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ by $k(x^*) := \inf_{u^* \in \mathbb{R}^p} \varphi^*(x^*, u^*)$, then $\beta = k(0)$.

This analysis is summarized in the following scheme:

$$\begin{array}{ll} \alpha = \inf f(x) & (P) \quad \beta = \inf h^*(u^*) \quad (Q) \\ \varphi(x, 0) = f(x), \forall x \in \mathbb{R}^n & \varphi^*(0, u^*) = h^*(u^*), \forall u^* \in \mathbb{R}^p \\ h(u) = \inf_x \varphi(x, u) & k(x^*) = \inf_{u^*} \varphi^*(x^*, u^*) \\ \alpha = h(0) & \beta = k(0) \end{array}$$

$$-\beta \leq \alpha.$$

If h is proper and convex, a necessary and sufficient condition for ensuring that there will be no duality gap ($-\beta = \alpha$) is that h be l.s.c. at 0 (in general φ l.s.c. does not imply that h would be l.s.c.).

Further more, if h is convex, l.s.c. and $0 \in \text{ri}(\text{dom}(h))$, then $\alpha = -\beta$ and the dual problem has at least one optimal solution, and if $\overline{u^*}$ is an optimal solution of (Q) and $\varphi = \varphi^{**}$, then

$$\overline{x} \text{ is an optimal solution of } (P) \iff f(\overline{x}) + h^*(\overline{u^*}) = 0.$$

Consider now the function $g : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \overline{\mathbb{R}}$ defined by:

$$g(x, u^*) = f(x) + h^*(u^*).$$

This function vanishes at (x_0, u_0^*) if and only if x_0 solves the primal problem and u_0^* solves the dual one. In addition, this function is non-negative and if the first variable is kept fixed, the function is convex and l.s.c. It is clear now, which properties are satisfied for many gap functions.

3 Generalized Conjugation

Definition 3.1 A non-negative function $g : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ will be called a G-coupling function, if there exists a non-empty closed set $C \subset \mathbb{R}^m$ such that:

- (D1) $\text{dom}(g) = \mathbb{R}^n \times C$, which means that $g : \mathbb{R}^n \times C \rightarrow \mathbb{R}$ and $g(x, x^*) = +\infty$ for every $x^* \notin C$.
- (D2) $\inf_{x \in \mathbb{R}^n, x^* \in C} g(x, x^*) = 0$.

Define $\mathcal{F}^{n,m} = \{g : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \overline{\mathbb{R}} : g \text{ is a G-coupling function}\}$.

Not every G-coupling function has zeros:

Example: Define on \mathbb{R}^2

$$g(x, x^*) = \exp(x + x^*).$$

Then $g \in \mathcal{F}^{1,1}$ ($C = \mathbb{R}$) is continuous and it does not have any zeros.

PROPOSITION 3.2 Let $g \in \mathcal{F}^{n,n}$ and $C \subset \mathbb{R}^n$ be the non-empty closed set linked to g . The following statements hold

- i) If $\lim_{\|(x, x^*)\| \rightarrow +\infty} g(x, x^*) = +\infty$, with $(x, x^*) \in \mathbb{R}^n \times C$ and g is l.s.c. then

$$\bigcap_{\varepsilon > 0} (S_g(\varepsilon) = \{(x, x^*) \in \mathbb{R}^n \times C : g(x, x^*) \leq \varepsilon\}) \neq \emptyset.$$

- ii) If g is lsc on $\mathbb{R}^n \times C$ and there exists $M > 0$ such that $S_g(M)$ is bounded, then

$$\{(x, x^*) : g(x, x^*) = 0\} = \bigcap_{\varepsilon > 0} S_g(\varepsilon) \neq \emptyset.$$

Proof

- i) It is well known (see [1]) that $\lim_{\|(x,x^*)\| \rightarrow +\infty} g(x,x^*) = +\infty$ is equivalent to the fact that all the level sets $S_g(\varepsilon)$ are bounded. The statement follows from the lsc of g .
- ii) It follows from the lsc of g .

□

Let us turn our attention now to how the family of functions $\mathcal{F}^{n,m}$ will allow us to establish duality schemes for, at least, the minimization problem. It is important to point out that in the following we consider an unusual type of duality: f is kept fixed and $g \in \mathcal{F}^{n,m}$, for a given $m \in \mathbb{N}$, is variable.

Consider a proper function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$. For a given $m \in \mathbb{N}$ take $g \in \mathcal{F}^{n,m}$. Define $f^g : C \rightarrow \mathbb{R} \cup \{+\infty\}$ and $f^{gg} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ as follows (for example see [9], [11] and references therein):

$$f^g(x^*) := \sup_{x \in \mathbb{R}^n} \{g(x, x^*) - f(x)\} \quad \forall x^* \in C,$$

$$f^{gg}(x) := \sup_{x^* \in C} \{g(x, x^*) - f^g(x^*)\} \quad \forall x \in \mathbb{R}^n,$$

where C is the closed set linked to g . In some cases, it would be better to consider a $g \in \mathcal{F}^{n,m}$ which satisfies:

- (D3) C is convex and $g(x, \cdot) : C \rightarrow \mathbb{R}$ is a convex and l.s.c. function for each x in \mathbb{R}^n .

With this, we have the following:

LEMMA 3.3 *Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper function and given $m \in \mathbb{N}$. If $g \in \mathcal{F}^{n,m}$, then*

$$f^g(x^*) + f(x) \geq g(x, x^*) \geq 0, \quad \forall (x, x^*) \in \mathbb{R}^n \times C,$$

which implies

$$f(x) \geq -f^g(x^*), \quad \forall (x, x^*) \in \mathbb{R}^n \times C.$$

Furthermore, if g satisfies (D3), then f^g is a convex l.s.c function.

Unless it is mentioned, not every $g \in \mathcal{F}^{n,m}$ satisfies (D3).

It would be interesting to know which condition either a G-coupling function g or the function f must satisfy in order that the function f^g be proper, because with this one would have a non-trivial function related to f . The following lemma ensures the existence of such a function $g \in \mathcal{F}^{n,m}$ for any

$m \in \mathbb{N}$, taking as a starting point a natural condition on f which must be imposed if f is the objective function of a minimization problem.

LEMMA 3.4 *Let f be as before. Then f is bounded from below if and only if, for every $m \in \mathbb{N}$, there exists $g \in \mathcal{F}^{n,m}$ such that f^g is proper.*

Proof

- Suppose that $\inf f > -\infty$, then for a $m_0 \in \mathbb{N}$ fixed, consider $g \in \mathcal{F}^{n,m_0}$ as follows:

$$g(x, x^*) = \begin{cases} \|x^*\| & \text{if } x \in \text{dom}(f) \\ 0 & \text{if } x \notin \text{dom}(f), \end{cases}$$

($C = \mathbb{R}^{m_0}$) thus

$$f^g(x^*) = \|x^*\| - \inf f \quad \forall x^* \in \mathbb{R}^{m_0},$$

which is clearly a proper function and since $m_0 \in \mathbb{N}$ was fixed arbitrarily, the result is satisfied for every $m \in \mathbb{N}$.

- Take $m_0 \in \mathbb{N}$ and $g \in \mathcal{F}^{n,m_0}$ such that f^g is proper. Let us suppose that $\inf f = -\infty$, from [13] we can see that this implies that $\inf f^{gg} = -\infty$. Then:

$$-\infty = \inf f^{gg} = \inf_{x \in \mathbb{R}^n} \left(\sup_{x^* \in C} [g(x, x^*) - f^g(x^*)] \right) \geq$$

$$\sup_{x^* \in C} \left(\inf_{x \in \mathbb{R}^n} [g(x, x^*) - f^g(x^*)] \right) \geq \sup_{x^* \in C} (-f^g(x^*)) = - \inf_{x^* \in C} f^g(x^*),$$

which means $-\infty \geq - \inf_{x^* \in C} (f^g(x^*))$. Then $\inf_{x^* \in C} f^g(x^*) = +\infty$, which implies that f^g is not proper and we have a contradiction. Therefore, $\inf f > -\infty$.

□

Notice that this proof also states, in particular, that there exists $g \in \mathcal{F}^{n,m}$ for every $m \in \mathbb{N}$ which satisfies (D3) and f^g is proper.

Henceforth, consider only functions f such that $\inf f > -\infty$ and for some fixed $m \in \mathbb{N}$, $g \in \mathcal{F}^{n,m}$ will be such that f^g is proper.

Let it be $\mathcal{F}^n = \{f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}, f \text{ is proper}, \inf f > -\infty\}$ and

$\gamma_{g,f} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by:

$$\gamma_{g,f}(x, x^*) = \begin{cases} f(x) + f^g(x^*) & \text{if } x^* \in C \\ +\infty & \text{otherwise.} \end{cases}$$

with $m \in \mathbb{N}$, $g \in \mathcal{F}^{n,m}$ and $f \in \mathcal{F}^n$. Define

$$\mathcal{F}_f^{n,m} := \{g \in \mathcal{F}^{n,m} / f^g \text{ is proper and } \inf \gamma_{g,f} = 0\},$$

observe that $\gamma_{g,f}$ might not be in $\mathcal{F}^{n,m}$, since $\gamma_{g,f}$ can take the value $+\infty$ for some $(x, x^*) \in \mathbb{R}^n \times C$.

LEMMA 3.5 $\mathcal{F}_f^{n,m}$ is non-empty for all $m \in \mathbb{N}$.

Proof Given $m \in \mathbb{N}$, define $g \in \mathcal{F}^{n,m}$ by:

$$g(x, x^*) = \begin{cases} \|x^*\| & \text{if } x \in \text{dom}(f) \\ 0 & \text{if } x \notin \text{dom}(f). \end{cases}$$

It is easy to check that g belongs to $\mathcal{F}_f^{n,m}$ with $C = \mathbb{R}^m$ (this example also proves that functions can be found in $\mathcal{F}_f^{n,m}$ which satisfy (D3)). \square

Now consider

$$(P) \min_x f(x)$$

with $f \in \mathcal{F}^n$. Taking $g \in \mathcal{F}_f^{n,m}$, define the dual problem related to g :

$$(D_g) \min_{x^* \in C} f^g(x^*).$$

Since

$$\inf_{(x, x^*) \in \mathbb{R}^n \times C} \gamma_{g,f}(x, x^*) = \inf_{x \in \mathbb{R}^n} f(x) + \inf_{x^* \in C} f^g(x^*) = 0$$

$$\implies \inf_{x \in \mathbb{R}^n} f(x) = - \inf_{x^* \in C} f^g(x^*) = \sup_{x^* \in C} [-f^g(x^*)].$$

This means that there is no duality gap between the primal problem (P) and its dual (D_g) for every $g \in \mathcal{F}_f^{n,m}$.

The next theorem states that given $n, m \in \mathbb{N}$, the correspondence defined by

$$\begin{aligned} \mathbf{F} : \mathcal{F}^n &\rightrightarrows \mathcal{F}^{n,m} \\ f &\mapsto \mathbf{F}(f) = \mathcal{F}_f^{n,m}, \end{aligned}$$

is a closed correspondence (see [17]).

THEOREM 3.6 *Take $f \in \mathcal{F}^n$ and $C \subset \mathbb{R}^m$ the non-empty closed convex set. If there exist $f_k : \text{dom}(f) \rightarrow \mathbb{R}$, $g_k : \mathbb{R}^n \times C \rightarrow \mathbb{R}$, sequences of functions ($k \in \mathbb{N}$), such that f_k converges uniformly to f (in $\text{dom}(f)$), $g_k \in \mathcal{F}_{f_k}^{n,m}$ satisfies (D3) for every $k \in \mathbb{N}$ and g_k converges uniformly to a function g (in $\mathbb{R}^n \times C$), then $g \in \mathcal{F}_f^{n,m}$ and it satisfies (D3) (extend g to $\mathbb{R}^n \times \mathbb{R}^m$ by only taking $g(x, x^*) = +\infty$ whenever $x^* \notin C$).*

Proof Let us prove first that $g \in \mathcal{F}^{n,m}$. Since g_k converges uniformly to g , given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $k \geq N$ then

$$|g_k(x, x^*) - g(x, x^*)| < \varepsilon, \quad \forall (x, x^*) \in \mathbb{R}^n \times C.$$

$$\text{Hence } g_k(x, x^*) - \varepsilon < g(x, x^*) < g_k(x, x^*) + \varepsilon, \quad \forall (x, x^*) \in \mathbb{R}^n \times C.$$

Taking $\inf_{x, x^* \in C}$ (remember that $\inf g_k = 0$ for all $k \in \mathbb{N}$):

$$-\varepsilon < \inf_{x, x^* \in C} g(x, x^*) < \varepsilon.$$

Then $|\inf g| < \varepsilon$. And since $\varepsilon > 0$ is arbitrary, one has that $\inf g = 0$.

Now we prove that $g(x, \cdot) : C \rightarrow \mathbb{R}$ is convex and l.s.c. for all $x \in \mathbb{R}^n$.

- $g(x, \cdot)$ is convex: let $x \in \mathbb{R}^n$ be fixed arbitrarily. Since for all $k \in \mathbb{N}$, $g_k(x, \cdot)$ is convex, one has that given $x_1^*, x_2^* \in C$ and $t \in [0, 1]$:

$$g_k(x, tx_1^* + (1-t)x_2^*) \leq tg_k(x, x_1^*) + (1-t)g_k(x, x_2^*).$$

Making $k \rightarrow +\infty$:

$$g(x, tx_1^* + (1-t)x_2^*) \leq tg(x, x_1^*) + (1-t)g(x, x_2^*),$$

which proves that $g(x, \cdot)$ is convex.

- $g(x, \cdot)$ is l.s.c.: take $\lambda < g(x, x^*)$. There exists $N \in \mathbb{N}$ such that

$$|g_N(y, y^*) - g(y, y^*)| < \varepsilon, \quad \forall (y, y^*) \in \mathbb{R}^n \times C,$$

where $\varepsilon = \frac{g(x, x^*) - \lambda}{2}$.

Hence $\lambda < \lambda + \varepsilon = g(x, x^*) - \varepsilon < g_N(x, x^*)$.

Since $g_N(x, \cdot)$ is l.s.c., then there exists $V(x^*) \subset C$, a neighborhood of x^* , such that if $y^* \in V(x^*)$ then

$$\lambda + \varepsilon < g_N(x, y^*).$$

Reducing $g(x, y^*)$:

$$\lambda + \varepsilon - g(x, y^*) < g_N(x, y^*) - g(x, y^*) < \varepsilon.$$

Therefore, if $y^* \in V(x^*)$, then $\lambda < g(x, y^*)$. Thus $g(x, \cdot)$ is l.s.c. for all $x \in \mathbb{R}^n$.

This proves that $g \in \mathcal{F}^{n,m}$.

It remains to prove that $g \in \mathcal{F}_f^{n,m}$. For doing this, let us show that $(f_k^{g_k})_{k \in \mathbb{N}}$ converges uniformly to f^g .

Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that if $k \geq N$ then

$$|g_k(x, x^*) - g(x, x^*)| < \frac{\varepsilon}{4}, \quad \forall (x, x^*) \in \mathbb{R}^n \times C \text{ and}$$

$$|f_k(x) - f(x)| < \frac{\varepsilon}{4}, \quad \forall x \in \mathbb{R}^n.$$

Fix $k \geq N$ and take $x^* \in C$ arbitrarily, then

$$f_k^{g_k}(x^*) - \frac{\varepsilon}{2} < g_k(x', x^*) - f_k(x'), \text{ for some } x' \in \mathbb{R}^n.$$

$$\text{Hence } f_k^{g_k}(x^*) - \varepsilon < g_k(x', x^*) - f_k(x') - \frac{\varepsilon}{2} < g(x', x^*) - f(x') \leq f^g(x^*),$$

$$\text{and so } f_k^{g_k}(x^*) - \varepsilon < f^g(x^*).$$

This proves that $f_k^{g_k}(x^*) - f^g(x^*) < \varepsilon$. On the other hand:

$$f^g(x^*) - \frac{\varepsilon}{2} < g(x'', x^*) - f(x''), \text{ for some } x'' \in \mathbb{R}^n,$$

whence $f^g(x^*) - \varepsilon < g(x'', x^*) - f(x') - \frac{\varepsilon}{2} < g_k(x'', x^*) - f_k(x'')$,

and so $f^g(x^*) - \varepsilon < f_k^{g_k}(x^*)$.

With this, it is shown that $-\varepsilon < f_k^{g_k}(x^*) - f^g(x^*)$ for an arbitrary $x^* \in C$. Therefore $(f_k^{g_k})_{k \in \mathbb{N}}$ converges uniformly to f^g (in C), and it is immediate to see that f^g is proper and

$$0 \leq f(x) + f^g(x^*) \leq f_k(x) + f_k^{g_k}(x^*) + \varepsilon, \quad \forall x \in \text{dom}(f), \quad x^* \in C$$

where $\varepsilon > 0$ is arbitrary and k is large enough. Taking $\inf_{(x, x^*) \in \mathbb{R}^n \times C}$ one has:

$$0 \leq \inf_{(x, x^*) \in \mathbb{R}^n \times C} (f(x) + f^g(x^*)) \leq \varepsilon.$$

Therefore $\inf_{(x, x^*) \in \mathbb{R}^n \times C} (f(x) + f^g(x^*)) = 0$ and $g \in \mathcal{F}_f^{n, m}$. \square

This theorem proves a more difficult situation, the case when $g_k \in \mathcal{F}_{f_k}^{n, m}$ satisfy (D3) for all $k \in \mathbb{N}$. For the general case, just omit the two \bullet items and change C for the non-empty closed set linked to g .

Consider now $g \in \mathcal{F}_f^{n, m}$. Define $l : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ as follows:

$$l(x^*) = \begin{cases} f^g(x^*) & \text{if } x^* \in C \\ +\infty & \text{if } x^* \notin C. \end{cases} \quad (1)$$

LEMMA 3.7 *Let $g \in \mathcal{F}_f^{n, m}$ satisfy (D3), l be defined like in (1) and $l^*(x) = \sup_{x^* \in \mathbb{R}^m} \{\langle x^*, x \rangle - l(x^*)\}$, if $0 \in \text{ri}(\text{dom}(l^*))$. Then the following statements are equivalent:*

- i) $\partial l^*(0) \neq \emptyset$.
- ii) (D_g) has a solution.

Proof

$$\partial l^*(0) \neq \emptyset \iff \exists \bar{x}^* \text{ such that } l^*(0) + l^{**}(\bar{x}^*) = l^*(0) + l(\bar{x}^*) = 0.$$

This is equivalent to

$$l(\bar{x}^*) = -l^*(0) = - \sup_{x^* \in \mathbb{R}^m} \{-l(x^*)\} = \inf_{x^* \in \mathbb{R}^m} l(x^*) = \inf_{x^* \in C} f^g(x^*),$$

but $\inf_{x^* \in C} f^g(x^*) = -\inf f \neq +\infty$ then $l(\bar{x}^*) \neq +\infty$, therefore $\bar{x}^* \in C$ and

$$l(\bar{x}^*) = f^g(\bar{x}^*) = \inf_{x^* \in C} f^g(x^*).$$

□

THEOREM 3.8 \bar{x}^* is a solution of (D_g) and \bar{x} is a solution of (P) if and only if $\gamma_{g,f}(\bar{x}, \bar{x}^*) = 0$.

Proof \bar{x} and \bar{x}^* are solutions of (P) and (D_g) respectively if and only if

$$f(\bar{x}) = \inf f = -\inf f^g = -f^g(\bar{x}^*) \iff f(\bar{x}) + f^g(\bar{x}^*) = \gamma_{g,f}(\bar{x}, \bar{x}^*) = 0.$$

□

Define $m(\gamma_{g,f}) := \{(x, x^*) \in \mathbb{R}^n \times C : \gamma_{g,f}(x, x^*) = 0\}$, for the previous theorem, (x_0, x_0^*) belongs to $m(\gamma_{g,f})$ if and only if x_0 is a solution of (P) and x_0^* is a solution of (D_g) . Take $f \in \mathcal{F}^n$ and $g \in \mathcal{F}_f^{n,m}$. Define the set $R(\gamma_{g,f})$, as follows:

$$R(\gamma_{g,f}) := \bigcap_{(x, x^*) \in \mathbb{R}^n \times C} (S_{\gamma_{g,f}(x, x^*)}(\gamma_{g,f}))^\infty,$$

where $S_\lambda(\gamma_{g,f})$ stands for the λ -level set of the function $\gamma_{g,f}$

$$S_\lambda(\gamma_{g,f}) := \{(x, x^*) : \gamma_{g,f}(x, x^*) \leq \lambda\}.$$

and for $A \subset \mathbb{R}^p$,

$$A^\infty := \{d \in \mathbb{R}^p : \exists \{x_k\}_{k \in \mathbb{N}} \subset A, \{t_k\}_{k \in \mathbb{N}} \downarrow 0 : t_k x_k \rightarrow d\},$$

with the convention, $\emptyset^\infty = \{0\}$ the following properties are satisfied (see [1]):

LEMMA 3.9 Let it be $A, B \subset \mathbb{R}^p$. The following are true:

- i) A^∞ is a closed cone.
- ii) $A^\infty = \{0\}$ if and only if A is bounded.
- iii) If $A \subset B$ then $A^\infty \subset B^\infty$.
- iv) $A^\infty = (\overline{A})^\infty$.
- v) If A is a cone, then $A^\infty = \overline{A}$.
- vi) If $(A_i)_{i \in I}$ is a family of sets in \mathbb{R}^p , we have that

$$\left(\bigcap_{i \in I} A_i \right)^\infty \subset \bigcap_{i \in I} A_i^\infty.$$

The proof of the next Lemma follows from: item (vi) of Lemma 3.9, the definition of $R(\gamma_{g,f})$ and the fact that $m(\gamma_{g,f}) = \bigcap_{(x,x^*) \in \mathbb{R}^n \times C} (S_{\gamma_{g,f}(x,x^*)}(\gamma_{g,f}))$.

LEMMA 3.10 $(m(\gamma_{g,f}))^\infty \subset R(\gamma_{g,f})$.

Note that, if $R(\gamma_{g,f}) = \{0\}$, then from Lemma 3.10 $m(\gamma_{g,f})$ is bounded (may be empty).

LEMMA 3.11 Let $\{\lambda_k\}_{k \in \mathbb{N}}$ be a sequence such that $\lim_{k \rightarrow +\infty} \lambda_k = 0$ and $\lambda_k > 0 \forall k \in \mathbb{N}$. If $m(\gamma_{g,f}) = \emptyset$, then $R(\gamma_{g,f}) = \bigcap_{k \in \mathbb{N}} (S_{\lambda_k}(\gamma_{g,f}))^\infty$.

Proof We only need to show that $\bigcap_{k \in \mathbb{N}} (S_{\lambda_k}(\gamma_{g,f}))^\infty \subset R(\gamma_{g,f})$. Indeed, take

$u \in \bigcap_{k \in \mathbb{N}} (S_{\lambda_k}(\gamma_{g,f}))^\infty$, then $u \in (S_{\lambda_k}(\gamma_{g,f}))^\infty \forall k \in \mathbb{N}$. For each $(x, x^*) \in \mathbb{R}^n \times C$ (arbitrarily fixed), we have that $0 < \gamma_{g,f}(x, x^*)$ (because $m(\gamma_{g,f}) = \emptyset$). Since $\lim_{n \rightarrow +\infty} \lambda_n = 0$, we have that there exists $q \in \mathbb{N}$ such that $\lambda_q \leq \gamma_{g,f}(x, x^*)$. So, $S_{\lambda_q}(\gamma_{g,f}) \subset S_{\gamma_{g,f}(x,x^*)}(\gamma_{g,f})$. It implies that $(S_{\lambda_k}(\gamma_{g,f}))^\infty \subset (S_{\gamma_{g,f}(x,x^*)}(\gamma_{g,f}))^\infty$ and so $u \in (S_{\gamma_{g,f}(x,x^*)}(\gamma_{g,f}))^\infty$. The statement follows. \square

For the next Lemma, consider that the function $\gamma_{g,f}$ is l.s.c. (it can be obtained, for example, if f and $g(x, \cdot) \forall x \in \mathbb{R}^n$ are l.s.c.).

LEMMA 3.12 $m(\gamma_{g,f}) = \emptyset$ if and only if $(m(\gamma_{g,f}))^\infty \neq R(\gamma_{g,f})$

Proof

\rightarrow If $m(\gamma_{g,f}) = \emptyset$, then $(m(\gamma_{g,f}))^\infty = \{0\}$ and $\gamma_{g,f}(x, x^*) > 0 \forall (x, x^*) \in \mathbb{R}^n \times C$. From, item (ii) of Proposition 3.2, we have that $(S_{\gamma_{g,f}(x,x^*)}(\gamma_{g,f}))$ are unbounded $\forall (x, x^*) \in \mathbb{R}^n \times C$. Now, consider $\{\lambda_k\}_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow +\infty} \lambda_k = 0$ and $\lambda_k > \lambda_{k+1} \forall k \in \mathbb{N}$. Take $u^k \in (S_{\lambda_k}(\gamma_{g,f}))^\infty$ with $\|u^k\| = 1 \forall k \in \mathbb{N}$. Since $\lambda_q \leq \lambda_k \forall k \in \mathbb{N}$ and $\forall q \geq k$, we have that $(S_{\lambda_q}(\gamma_{g,f}))^\infty \subset (S_{\lambda_k}(\gamma_{g,f}))^\infty$ and $\{u^q\}_{q \geq k} \subset (S_{\lambda_k}(\gamma_{g,f}))^\infty \forall k \in \mathbb{N}$ and so any cluster point of $\{u^k\}_{k \in \mathbb{N}}$ belong to $(S_{\lambda_k}(\gamma_{g,f}))^\infty \forall k \in \mathbb{N}$. From Lemma 3.11 we have that any cluster point of $\{u^k\}_{k \in \mathbb{N}}$ belong to $R(\gamma_{g,f})$. So, the statement follows.

\leftarrow It is equivalent to show that: If $m(\gamma_{g,f}) \neq \emptyset$, then $R(\gamma_{g,f}) = (m(\gamma_{g,f}))^\infty$. Indeed, we know that $S_0(\gamma_{g,f}) = m(\gamma_{g,f}) \neq \emptyset$. So, we have that, $R(\gamma_{g,f}) \subset (S_0(\gamma_{g,f}))^\infty = (m(\gamma_{g,f}))^\infty$. The statement follows.

\square

THEOREM 3.13 Given $n, m \in \mathbb{N}$, $f \in \mathcal{F}^n$ l.s.c. and $g \in \mathcal{F}_f^{n,m}$ such that $g(x, \cdot)$

is l.s.c. for all $x \in \mathbb{R}^n$. The following statements are equivalents.

- (i) $R(\gamma_{g,f}) = \{0\}$.
- (ii) $m(\gamma_{g,f})$ is nonempty and compact.

Proof We need to proof only the implication **(i) implies (ii)**. Indeed, from l.s.c of f and $g(x, \cdot) \forall x \in \mathbb{R}^n$, we have that $\gamma_{g,f}$ is l.s.c. on $\mathbb{R}^n \times C$. Since $R(\gamma_{g,f}) = \{0\}$, then from Lemma 3.10 $(m(\gamma_{g,f}))^\infty = \{0\}$. So, from Lemma 3.12 we have $m(\gamma_{g,f}) \neq \emptyset$, here $m(\gamma_{g,f})$ is closed. The statement follows applying Lemma 3.9 item (ii). \square

At this point a natural question arises, if $g \in \mathcal{F}_f^{n,m}$, would be there any kind of relation between the optimal points and the optimal values of f and f^{gg} ? The next lemma answers this.

LEMMA 3.14 *For a fixed $m \in \mathbb{N}$ and every $g \in \mathcal{F}_f^{n,m}$, the following are satisfied:*

- i) $\inf f = \inf f^{gg}$,
- ii) if x_0 is a global minimum of f , then x_0 is a global minimum of f^{gg} .

Proof Remember that f^{gg} is defined by:

$$f^{gg}(x) = \sup_{x^* \in C} \{g(x, x^*) - f^g(x^*)\},$$

where C is the closed set linked to g .

- i) $\inf f^{gg} \leq \inf f$ is always true. On the other hand

$$f^g(x^*) + f^{gg}(x) \geq g(x, x^*) \geq 0, \forall x \in \mathbb{R}^n, x^* \in C,$$

which implies that

$$\inf f^{gg} \geq - \inf_{x^* \in C} f^g(x^*).$$

But, since $g \in \mathcal{F}_f^{n,m}$ one has that

$$\inf f = - \inf_{x^* \in C} f^g(x^*),$$

which means

$$\inf f \leq \inf f^{gg} \leq \inf f.$$

Therefore $\inf f = \inf f^{gg}$.

ii) $f^{gg}(x_0) \leq f(x_0) = \inf f = \inf f^{gg} \leq f^{gg}(x_0)$, then $f^{gg}(x_0) = \inf f^{gg}$.

□

It would be interesting to get a duality scheme which generalizes others like the Lagrangian one. By example, let us consider

$$g(x, x^*) = \begin{cases} f(x) + h^*(x^*) & \text{if } x \in \text{dom}(f), x^* \in \text{dom}(h^*) \\ 0 & \text{if } x \notin \text{dom}(f), x^* \in \text{dom}(h^*) \\ +\infty & \text{if } x^* \notin \text{dom}(h^*) \end{cases}$$

where h^* is the objective function of problem (Q) (section 2) and if in addition h is l.s.c. at 0, one has that $g \in \mathcal{F}_f^{n,p}$ (with $C = \text{dom}(h^*)$), even more $f^g \equiv h^*$:

$$f^g(x^*) = \sup_{x \in \mathbb{R}^n} \{g(x, x^*) - f(x)\}, \quad x^* \in C,$$

but

$$g(x, x^*) - f(x) = \begin{cases} f(x) + h^*(x^*) - f(x) & \text{if } x \in \text{dom}(f) \\ -\infty & \text{if } x \notin \text{dom}(f) \end{cases} \implies$$

$$g(x, x^*) - f(x) = \begin{cases} h^*(x^*) & \text{if } x \in \text{dom}(f) \\ -\infty & \text{if } x \notin \text{dom}(f), \end{cases}$$

thus $f^g \equiv h^*$ and the classic duality is recovered. (Later on, we will exhibit a more interesting $g \in \mathcal{F}_f^{n,p}$, at least in the case of a restricted problem.)

Examples: In the following examples, $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g \in \mathcal{F}^{1,1}$.

1) $f(x) = x^2$ and $g(x, x^*) = (xx^*)^2$ ($C = \mathbb{R}$). Calculate f^g :

$$f^g(x^*) = \sup_{x \in \mathbb{R}} \{(xx^*)^2 - x^2\} = \begin{cases} 0 & \text{if } |x^*| \leq 1 \\ +\infty & \text{if } |x^*| > 1. \end{cases}$$

Then

$$g'(x, x^*) = f(x) + f^g(x^*) = \begin{cases} x^2 & \text{if } |x^*| \leq 1 \\ +\infty & \text{if } |x^*| > 1 \end{cases}$$

and $g' \in \mathcal{F}_f^{1,1}$.

2) Take now

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ +\infty & \text{if } x < 0 \end{cases}$$

and

$$g(x, x^*) = \begin{cases} \frac{1}{xx^*+1} & \text{if } x, x^* \geq 0 \\ 0 & \text{if } x < 0, x^* \geq 0 \\ +\infty & \text{otherwise.} \end{cases}$$

$$C = [0, +\infty[,$$

$$f^g(x^*) = \sup_{x \in \mathbb{R}} \{g(x, x^*) - f(x)\} = \sup_{x \geq 0} \left\{ \frac{1}{xx^*+1} - x^2 \right\}, \quad x^* \in C$$

$$\implies f^g(x^*) = 1, \quad x^* \in C.$$

Then

$$g'(x, x^*) = f(x) + f^g(x^*) = \begin{cases} x^2 + 1 & \text{if } x, x^* \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

and $g \notin \mathcal{F}_f^{1,1}$.

3) Let $f(x) = \exp(x)$ and $g(x, x^*) = \exp(x + x^*)$ ($C = \mathbb{R}$). Then

$$f^g(x^*) = \sup_{x \in \mathbb{R}} \{\exp(x)(\exp(x^*) - 1)\} = \begin{cases} 0 & \text{if } x^* \leq 0 \\ +\infty & \text{if } x^* > 0. \end{cases}$$

And

$$g'(x, x^*) = f(x) + f^g(x^*) = \begin{cases} \exp(x) & \text{if } x^* \leq 0 \\ +\infty & \text{if } x^* > 0. \end{cases}$$

Thus $g \in \mathcal{F}_f^{1,1}$. \square

Now, take two particular $g \in \mathcal{F}^{n,n}$ which have some good properties (see [11]).

Firstly, consider $f \in \mathcal{F}^n$ such that $\text{dom}(f) \subset \mathbb{R}_+^n$ then take $g \in \mathcal{F}^{n,n}$ defined as follows:

$$g(x, x^*) := \begin{cases} \langle x^*, x \rangle^+ & x \in \mathbb{R}_+^n, x^* \in \mathbb{R}_+^n \\ 0 & x \notin \mathbb{R}_+^n, x^* \in \mathbb{R}_+^n \\ +\infty & x^* \notin \mathbb{R}_+^n, \end{cases}$$

where $\langle x^*, x \rangle^+ = \max_{i=1, \dots, n} \{x_i^* x_i\}$ (here $C = \mathbb{R}_+^n$). It is easy to prove that this function belongs to $\mathcal{F}_f^{n,n}$ and also satisfies the following:

- (*) $g(\cdot, x^*) : \mathbb{R}_+^n \rightarrow \mathbb{R}$ and $g(x, \cdot) : \mathbb{R}_+^n \rightarrow \mathbb{R}$ are convex, l.s.c. and IPH (increasing positively homogeneous) functions (with $x_1 \leq x_2$ if and only if $x_{1i} \leq x_{2i}$ for all $i = 1, \dots, n$) for every $x^*, x \in \mathbb{R}_+^n$.

Consider now the set of functions H_g defined as:

$$H_g := \{h : \mathbb{R}^n \rightarrow \mathbb{R} \text{ such that } \exists(x^*, \lambda) \in \mathbb{R}_+^n \times \mathbb{R} \text{ with } h(x) = g(x, x^*) - \lambda\}.$$

With this notation, define the support set of f with respect to H_g , $\text{supp}(f, H_g)$:

$$\text{supp}(f, H_g) := \{h \in H_g : h(x) \leq f(x) \forall x \in \mathbb{R}^n\}.$$

It is already proven (see [11]) that, since every $h \in H_g$ is linked to one and only one $(x^*, \lambda) \in \mathbb{R}_+^n \times \mathbb{R}$, we have

$$\text{supp}(f, H_g) = \text{epi}(f^g).$$

Let us see what properties are satisfied by f^{gg} and $\text{supp}(f, H_g)$.

- $f^{gg}(x) = \sup_{x^* \in \mathbb{R}_+^n} [g(x, x^*) - f^g(x^*)] \implies$

$$f^{gg}(x) = \begin{cases} \sup_{x^* \in \mathbb{R}_+^n} [\langle x^*, x \rangle^+ - f^g(x^*)] & x \in \mathbb{R}_+^n, \\ \inf_{y \in \mathbb{R}^n} f(y) & x \notin \mathbb{R}_+^n, \end{cases}$$

(recall that $\inf_{y \in \mathbb{R}^n} f(y) = \sup_{x^* \in \mathbb{R}_+^n} [-f^g(x^*)]$). Since the functions $\langle x^*, \cdot \rangle^+ - f^g(x^*)$ are increasing convex and l.s.c. for every $x^* \in \mathbb{R}_+^n$, then f^{gg} is increasing convex and l.s.c. on \mathbb{R}_+^n .

- $f^g(x^*) = \sup_{x \in \mathbb{R}^n} [g(x, x^*) - f(x)] = \sup_{x \in \text{dom}(f)} [\langle x^*, x \rangle^+ - f(x)]$ as before, the functions $\langle \cdot, x^* \rangle^+ - f(x)$ are convex increasing and l.s.c for every

$x \in \text{dom}(f) \subset \mathbb{R}_+^n$, therefore f^g is a convex increasing and l.s.c. function. With this, one has that $\text{supp}(f, H_g) = \text{epi}(f^g)$ is closed, convex and if f is non-negative then for every $(x^*, \lambda) \in \text{supp}(f, H_g)$, $t(x^*, \lambda) \in \text{supp}(f, H_g)$ for all $t \in [0, 1]$.

Thanks to [11] another interesting property can also be shown. Since \mathbb{R}_+^n is a convex closed cone, one has that for all $x \in \mathbb{R}_+^n$,

$$\partial f_{x_i}^{gg}(1) \neq \emptyset, \forall i = 1, \dots, n$$

where x_i is a vector with its components equal to zero except the i -th component which is equal to the i -th component of x and $f_{x_i}^{gg} : [0, +\infty] \rightarrow \mathbb{R}_{+\infty}$ is defined by

$$f_{x_i}^{gg}(t) := f^{gg}(tx_i), \forall t \in [0, +\infty].$$

Secondly, if we consider

$$g(x, x^*) := \begin{cases} \langle x^*, x \rangle^- & x \in \mathbb{R}_+^n, x^* \in \mathbb{R}_+^n \\ 0 & x \notin \mathbb{R}_+^n, x^* \in \mathbb{R}_+^n \\ +\infty & x^* \notin \mathbb{R}_+^n, \end{cases}$$

where $\langle x^*, x \rangle^- = \min_{i \in I_+(x^*)} \{x_i^* x_i\}$, $I_+(x^*) = \{i : l_i > 0\}$ (here also $C = \mathbb{R}_+^n$), then f^{gg} is a l.s.c. increasing convex-along-rays (ICAR) function ([11]) and we can find in chapter 3, section 3 of [11] several results about this kind of function including a condition which guarantees that in some points, the general sub-differential of the function f^{gg} is nonempty.

Remark: The previous results are valid for every $g \in \mathcal{F}_f^{n,n}$ which satisfies (*).

3.1 Particular Case: Classical Lagrangian Duality

Let

$$(P) : \min_{x \in A} f(x)$$

be a typical minimization problem, where

$$A := \{x \in \mathbb{R}^n : h_i(x) \leq 0, \forall i = 1, \dots, m\},$$

$f : A \rightarrow \mathbb{R}$, $h_i : \mathbb{R}^n \rightarrow \mathbb{R}$, are convex l.s.c functions with $i = 1, \dots, m$.

Remember that (see [6]) the following is the well known dual problem:

$$(D_L) : \min_{\lambda^* \geq 0} \sup_{x \in A} \{\langle \lambda^*, -h(x) \rangle - f(x)\},$$

$h(x) = (h_1(x), \dots, h_m(x))$. Moreover, x_0 is a solution of (P) and λ_0^* is a solution of (D_L) if and only if (x_0, λ_0^*) is a saddle point of the Lagrangian function L , given by

$$L(x, \lambda^*) := f(x) + \langle \lambda^*, h(x) \rangle,$$

which means,

$$L(x_0, \lambda^*) \leq L(x_0, \lambda_0^*) \leq L(x, \lambda_0^*), \quad \forall x \in A, \quad \forall \lambda^* \geq 0.$$

Define now $g_1 : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$, as follows:

$$g_1(y, \lambda^*) = \begin{cases} \langle \lambda^*, y \rangle & \text{if } y \geq 0, \lambda^* \geq 0 \\ 0 & \text{if } y \not\geq 0, \lambda^* \geq 0 \\ +\infty & \text{if } \lambda^* \not\geq 0. \end{cases}$$

It can be seen that $g_1 \in \mathcal{F}^{m,m}$, with $C = \mathbb{R}_+^m$. Let

$$g(x, \lambda^*) = g_1(-h(x), \lambda^*),$$

then $g \in \mathcal{F}^{n,m}$. Calculating $\bar{f}^g(\lambda^*)$ with $\lambda^* \in C$ and

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in A \\ +\infty & \text{if } x \notin A. \end{cases}$$

$$\text{Then } \bar{f}^g(\lambda^*) = \sup_x \{g(x, \lambda^*) - \bar{f}(x)\} = \sup_{x \in A} \{\langle \lambda^*, -h(x) \rangle - f(x)\},$$

which means

$$\bar{f}^g(\lambda^*) = \sup_{x \in A} \{\langle \lambda^*, -h(x) \rangle - f(x)\}, \text{ for every } \lambda^* \in C = \mathbb{R}_+^m$$

and thus the classical Lagrangian duality is recovered.

4 The Equilibrium Problem

Let $f : K \times \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$, where $K \subset \mathbb{R}^n$ is a non-empty closed convex set, be such that

- i) $f(x, \cdot) : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is a convex l.s.c. function for every $x \in K$.
- ii) $f(x, x) = 0, \forall x \in K$.

The Equilibrium Problem is defined as follows:

$$(EP) : \text{Find } x \in K \text{ such that } f(x, y) \geq 0, \forall y \in K.$$

Pseudo-monotone functions, which are defined as follows, are consider in [13]. A function $\varphi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ would be called pseudo-monotone if for every $x, y \in \mathbb{R}^n$ with $\varphi(x, y) \geq 0$, we have $\varphi(y, x) \leq 0$.

PROPOSITION 4.1 *Let $g \in \mathcal{F}^{n,n}$ and $C \subset \mathbb{R}^n$ be the non-empty closed set linked to g . If g is pseudo-monotone, then $g(x, y) = 0, \forall (x, y) \in C \times C$.*

Proof Since $g \in \mathcal{F}^{n,n}$, then $\forall (x, y) \in C \times C, g(x, y) \geq 0$ and $g(y, x) \geq 0$. Therefore if g is pseudo-monotone, it can be said that $g(y, x) \leq 0$ and $g(y, x) = 0, \forall (x, y) \in C \times C$. \square

Observe that this proposition affirms that the only pseudo-monotone G-coupling function is the null function.

Take now $m \in \mathbb{N}$ and $g \in \mathcal{F}^{n,m}$. Consider now for every $x \in K$:

$$(P^x) : \inf_{y \in \mathbb{R}^n} \overline{f}(x, y) \quad (D_g^x) : \inf_{x^* \in C} \overline{f}_x^g(x^*)$$

where $\overline{f}_x^g(x^*) = \sup_{y \in \mathbb{R}^n} \{g(y, x^*) - \overline{f}(x, y)\}$ for every $x^* \in C$ (C is the non-empty closed set linked to g) and

$$\overline{f}(x, y) = \begin{cases} f(x, y) & y \in K \\ +\infty & y \notin K. \end{cases}$$

It would be interesting to know if there exist $m \in \mathbb{N}$ and a $g \in \mathcal{F}^{n,m}$ which satisfy the *Zero Duality Gap Property* (ZDGP):

$$\inf_{y \in \mathbb{R}^n} \overline{f}(x, y) + \inf_{x^* \in C} \overline{f}_x^g(x^*) = 0, \forall x \in F$$

where

$$F := \left\{ x \in K : \inf_{y \in K} f(x, y) \neq -\infty \right\}.$$

(Notice that if F is empty, (EP) would have no solutions.) If there exists such a g , the following are equivalent:

- \bar{x} is a solution of (EP) .
- $\inf_{y \in \mathbb{R}^n} \bar{f}(\bar{x}, y) = 0$.
- There exists $\bar{x} \in K$ such that $\inf_{x^* \in C} \bar{f}_x^g(x^*) = 0$.

LEMMA 4.2 *There exists $g \in \mathcal{F}^{n,n}$ which satisfies the ZDGP.*

Proof Define $g \in \mathcal{F}^{n,n}$ by:

$$g(y, x^*) = \begin{cases} \langle x^*, y \rangle & \text{if } x^* \in K^+, y \in K \\ 0 & \text{if } x^* \in K^+, y \notin K \\ +\infty & x^* \notin K^+, \end{cases}$$

where $K^+ := \{x^* \in \mathbb{R}^n : \langle x^*, x \rangle \geq 0, \forall x \in K\}$, in this case $C = K^+$. Calculate $\bar{f}_x^g(x^*)$, with $x \in F$, $x^* \in C$:

$$\bar{f}_x^g(x^*) = \sup_{y \in K} \{\langle x^*, y \rangle - f(x, y)\}.$$

It is clear that

$$\inf_{x^* \in C} \bar{f}_x^g(x^*) \leq \bar{f}_x^g(0) = - \inf_{y \in \mathbb{R}^n} \bar{f}(x, y) = - \inf_{y \in K} f(x, y),$$

but for every $x^* \in C$, $y \in \mathbb{R}^n$ one has that $\bar{f}_x^g(x^*) + \bar{f}(x, y) \geq 0$, which implies that we always have $\inf_{x^* \in C} \bar{f}_x^g(x^*) \geq - \inf_{y \in \mathbb{R}^n} \bar{f}(x, y)$. Therefore

$$\inf_{x^* \in C} \bar{f}_x^g(x^*) = - \inf_{y \in \mathbb{R}^n} \bar{f}(x, y) \implies \inf_{x^* \in C} \bar{f}_x^g(x^*) + \inf_{y \in \mathbb{R}^n} \bar{f}(x, y) = 0.$$

Finally, there exists $g \in \mathcal{F}^{n,n}$ which satisfies the ZDGP. \square

Let us give now another function $g \in \mathcal{F}^{n,n}$ which also satisfies the ZDGP. In this case, this g will generate a duality scheme which has been already studied in [7].

Let $i_K : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be defined by $i_K(x^*) := \inf_{x \in K} \langle x^*, x \rangle$ and $K^* := \{x^* \in \mathbb{R}^n : i_K(x^*) > -\infty\}$, the effective domain of i_K . (Since i_K is a concave u.s.c

function, then K^* is a closed convex set.) Define then:

$$g(y, x^*) = \begin{cases} \langle x^*, y \rangle - i_K(x^*), & \text{if } x^* \in K^*, y \in K \\ 0 & \text{if } x^* \in K^*, y \notin K \\ +\infty & \text{if } x^* \notin K^*, \end{cases}$$

in this case $C = K^*$. Calculate now $\bar{f}_x^g(x^*)$ for $x \in F$:

$$\begin{aligned} \bar{f}_x^g(x^*) &= \sup_{y \in \mathbb{R}^n} [g(y, x^*) - \bar{f}(x, y)] \\ &= \sup_{y \in K} [\langle x^*, y \rangle - i_K(x^*) - f(x, y)] \\ &= \sup_{y \in K} [\langle x^*, y \rangle - f(x, y)] - i_K(x^*), \end{aligned}$$

but $\bar{f}_x^*(x^*) = \sup_{y \in \mathbb{R}^n} [\langle x^*, y \rangle - \bar{f}(x, y)] = \sup_{y \in K} [\langle x^*, y \rangle - f(x, y)]$, then

$$\bar{f}_x^g(x^*) = \bar{f}_x^*(x^*) - i_K(x^*).$$

PROPOSITION 4.3 *The function g defined above, satisfies the ZDGP.*

Proof Similar to Lemma 4.2. □

In [7] a very interesting result can be found.

THEOREM 4.4 *\bar{x} is a solution of (EP) if and only if, there exists $x^* \in K^*$ such that $\bar{f}_{\bar{x}}^*(x^*) - i_K(x^*) = 0$.*

This result does not only say that $\inf_{x^* \in K^*} \bar{f}_{\bar{x}}^g(x^*) = 0$, but that the dual problem $(D_{\bar{g}}^{\bar{x}})$ has a solution. In order to prove this, we need first the following lemma.

LEMMA 4.5 *For every $x^* \in K^*$ and $x \in F$, one has:*

$$\bar{f}_x^*(x^*) - i_K(x^*) \geq 0.$$

Proof From Fenchel's inequality, we have for every $x \in F$ fixed

$$\langle x^*, y \rangle - \bar{f}_x^*(x^*) \leq \bar{f}(x, y), \quad \forall x^* \in K^*, \quad \forall y \in \mathbb{R}^n.$$

Then, taking $\inf_{y \in K}$:

$$i_K(x^*) - \bar{f}_x^*(x^*) \leq \inf_{y \in K} f(x, y) \leq 0,$$

where the last inequality occurs, since $\inf_{y \in K} f(x, y) \leq f(x, x) = 0$. This means,
 $\bar{f}_x^*(x^*) - i_K(x^*) \geq 0$. \square

Proof of Theorem 4.4:

- If \bar{x} is a solution of (EP), then

$$\bar{f}_{\bar{x}}(\bar{x}) = f_{\bar{x}}(\bar{x}) = 0 = \min_{y \in K} \bar{f}(\bar{x}, y) = \min_{y \in K} f(\bar{x}, y).$$

Then, there exists (see [10]) $x^* \in \partial f_{\bar{x}}(\bar{x}) \cap (-N_K(\bar{x}))$ (where $N_K(\bar{x})$ stands for the normal cone of K at \bar{x}) and thus

$$\bar{f}_{\bar{x}}^*(x^*) \leq f_{\bar{x}}^*(x^*) = f_{\bar{x}}(\bar{x}) + f_{\bar{x}}^*(x^*) = \langle x^*, \bar{x} \rangle = i_K(x^*),$$

which means $\bar{f}_{\bar{x}}^*(x^*) - i_K(x^*) \leq 0$. But thanks to the previous lemma, this implies that $\bar{f}_{\bar{x}}^*(x^*) - i_K(x^*) = 0$.

- Take $y \in K$ and suppose that there exists $x^* \in K^*$ such that $\bar{f}_{\bar{x}}^*(x^*) = i_K(x^*)$, then

$$f(\bar{x}, y) = \bar{f}(\bar{x}, y) \geq \langle x^*, y \rangle - \bar{f}_{\bar{x}}^*(x^*) = \langle x^*, y \rangle - i_K(x^*) \geq 0.$$

And thus, \bar{x} is a solution of (EP). \square

4.1 Particular Case: The Complementarity Problem

A particular case of the Equilibrium Problem is the Complementarity Problem, which is defined as follows:

$$(CP) : \text{Find } x \in K \text{ such that } T(x) \in K^+ \text{ and } \langle T(x), x \rangle = 0,$$

where $K \subset \mathbb{R}^n$ is a closed convex cone and $T : K \rightarrow \mathbb{R}^n$ is a continuous function. Considering $f(x, y) = \langle T(x), y - x \rangle$ with x and y in K , the solution set of the (CP) is equal to the solution set of (EP) related to f .

Let us take $g \in \mathcal{F}^{n,n}$ defined as in the beginning of this section:

$$g(y, x^*) = \begin{cases} \langle x^*, y \rangle & \text{if } x^* \in K^+, y \in K \\ 0 & \text{if } x^* \in K^+, y \notin K \\ +\infty & x^* \notin K^+. \end{cases}$$

Calculate $\bar{f}_x^g(x^*)$ ($x \in K$, $x^* \in C = K^+$):

$$\bar{f}_x^g(x^*) = \sup_{y \in \mathbb{R}^n} \{g(y, x^*) - \bar{f}(x, y)\} = \sup_{y \in K} \{g(y, x^*) - f(x, y)\}$$

$$\Rightarrow \bar{f}_x^g(x^*) = \sup_{y \in K} \{\langle x^* - T(x), y \rangle\} + \langle T(x), x \rangle$$

$$\Rightarrow \bar{f}_x^g(x^*) = \begin{cases} \langle T(x), x \rangle & x^* - T(x) \in K^- \\ +\infty & \text{otherwise.} \end{cases}$$

But $x^* - T(x) \in K^-$ is equivalent to the statement that

$$x \in T^{-1}(x^* + K^+) \subset T^{-1}(K^+),$$

and this inclusion is true since K is a closed convex cone and $x^* \in K^+$. Then

$$\bar{f}_x^g(x^*) = \begin{cases} \langle T(x), x \rangle & x \in T^{-1}(x^* + K^+) \\ +\infty & \text{otherwise.} \end{cases}$$

Therefore,

$$\bar{f}(x, y) + \bar{f}_x^g(x^*) = \begin{cases} \langle T(x), y \rangle & \text{if } x \in T^{-1}(x^* + K^+), y \in K \\ +\infty & \text{otherwise.} \end{cases}$$

Calculate now the set F :

- If $x \in K$ is such that $T(x) \in K^+$ then

$$\inf_{y \in K} f(x, y) = \inf_{y \in K} \langle T(x), y - x \rangle = -\langle T(x), x \rangle \neq -\infty,$$

which means $x \in F$.

- If $x \in K$ is such that $T(x) \notin K^+$ then there exists $y_0 \in K$ satisfying $\langle T(x), y_0 \rangle < 0$. Thus

$$\lim_{n \rightarrow +\infty} f(x, ny_0) = -\infty = \inf_{y \in K} f(x, y),$$

which means $x \notin F$.

All these imply that $F = T^{-1}(K^+)$.

Then, given $x_0 \in F$, there exists $x^* \in K^+$ (for example $x^* = 0$) such that $x_0 \in T^{-1}(x^* + K^+)$ and:

$$\inf_{y \in \mathbb{R}^n} \bar{f}(x_0, y) + \inf_{x^* \in C} \bar{f}_{x_0}^g(x^*) = 0,$$

but $x_0 \in F$ was chosen arbitrarily, therefore

$$\inf_{y \in \mathbb{R}^n} \bar{f}(x, y) + \inf_{x^* \in C} \bar{f}_x^g(x^*) = 0, \quad \forall x \in F.$$

Finally, there exists $\bar{x} \in K$ such that

$$\inf_{x^* \in C} \bar{f}_{\bar{x}}^g(x^*) = 0 \iff T(\bar{x}) \in K^+ \text{ and } \langle T(\bar{x}), \bar{x} \rangle = 0. \quad (2)$$

In [5] the (CP) is considered, when $K = K^+ = \mathbb{R}_+^n$ and T is an affine operator, in other words, the case of the Linear Complementarity Problem (LCP) . For studying this problem, they propose the following:

\bar{x} is a solution of (LCP) , if and only if \bar{x} satisfies:

$$\bar{x} \in \mathbb{R}_+^n, \quad T(\bar{x}) \in \mathbb{R}_+^n, \text{ and } \langle T(\bar{x}), \bar{x} \rangle = 0.$$

It is immediate to see that this proposal is identical to (2), therefore by using this $g \in \mathcal{F}^{n,n}$ (the one used at the beginning of this section) we generate a dual problem of (LCP) which has been treated in other works.

Conclusions

This work gives a basis for a new theory, which we called G-coupling functions. Logically there are many things to explore, by example:

- Using G-coupling functions for a Perturbation Theory.
- Using G-coupling functions for generating primal, dual and primal-dual algorithms.
- Analyze, using G-coupling functions, the Variational Inequality Problem for the case of non-monotone operators.
- Given f look for a G-coupling function such that f is abstract convex with respect to the class of elementary function induced G-coupling function (see [11]).

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